

# Charged particles in a rotating magnetic field\*

Qiong-gui Lin<sup>†</sup>

*China Center of Advanced Science and Technology (World Laboratory),*

*P.O.Box 8730, Beijing 100080, People's Republic of China<sup>‡</sup>*

*and*

*Department of Physics, Zhongshan University, Guangzhou 510275,*

*People's Republic of China*

We study the valence electron of an alkaline atom or a general charged particle with arbitrary spin and with magnetic moment moving in a rotating magnetic field. By using a time-dependent unitary transformation, the Schrödinger equation with the time-dependent Hamiltonian can be reduced to a Schrödinger-like equation with a time-independent effective Hamiltonian. Eigenstates of the effective Hamiltonian correspond to cyclic solutions of the original Schrödinger equation. The nonadiabatic geometric phase of a cyclic solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis, regardless of whether the solution is explicitly available. For the alkaline atomic electron and a strong magnetic field, the eigenvalue problem of the effective Hamiltonian is completely solved, and the geometric phase turns out to be a linear combination of two solid angles. For a weak magnetic field, the same problem is solved partly. For a general charged particle, the problem is solved approximately in a slowly rotating magnetic field, and the geometric phases are also calculated.

PACS number(s): 03.65.Ta

---

\*published in Phys. Rev. A **63** (2001) 012108.

<sup>†</sup>E-mail addresses: [qg\\_lin@163.net](mailto:qg_lin@163.net), [qg\\_lin@263.net](mailto:qg_lin@263.net)

<sup>‡</sup>not for correspondence

## I. INTRODUCTION

In quantum mechanics, the Schrödinger equation, even with a time-independent Hamiltonian, can be solved analytically only in a few cases. With a time-dependent Hamiltonian, the problem is more difficult and less examples are well studied. One of the well-studied examples is a nonrelativistic neutral particle with spin and magnetic moment in a rotating magnetic field [1-5]. This simple example has received much attention because of its relevance to the problem of geometric phases [6-8]. As an ordinary physical problem, it may be of equal interest to consider a charged particle in a rotating magnetic field. It appears to us that this problem was not considered in the literature. Because the problem can be treated analytically and some exact solutions are available, we will deal with it in this paper.

We will consider an electron moving in a central potential plus a strong rotating magnetic field in Sec. II. This can describe the valence electron of an alkaline atom or the one of a hydrogen atom under the influence of the external magnetic field. In Sec. III a weak rotating magnetic field is considered. In Sec. IV we consider a general charged particle with arbitrary spin and with magnetic moment moving in a rotating magnetic field, without the central potential. The Hamiltonians for all these systems are time dependent. We use a time-dependent unitary transformation to reduce the Schrödinger equation to a Schrödinger-like one with a time-independent effective Hamiltonian. Eigenstates of the effective Hamiltonian correspond to cyclic solutions of the original Schrödinger equation. The nonadiabatic geometric phase of a cyclic solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis, regardless of whether the solution is explicitly available. For the valence electron and a strong magnetic field considered in Sec. II, the eigenvalue problem of the effective Hamiltonian can be completely solved, and the geometric phase is obtained as a linear combination of two solid angles, of which one is subtended by the trace of the orbit angular momentum and the other by that of the spin. For a weak magnetic field in Sec. III, the spin-orbit coupling has to be included. This makes the effective Hamiltonian more complicated, and the eigenvalue problem can only be solved partly. In Sec. IV a scalar potential is absent but a quadratic term of the magnetic field cannot be discarded. Thus the problem in this section is rather different. For a slowly rotating magnetic field, one of the terms in the effective Hamiltonian can be treated as a small perturbation and the eigenvalue problem of the remaining terms can be solved exactly. In this approximation, the geometric phases of the cyclic solutions can be calculated explicitly and the results take a form similar to that in Sec. II. We give a brief summary and some discussions in Sec. V.

## II. ALKALINE ATOMIC ELECTRON IN A STRONG ROTATING MAGNETIC FIELD

Consider the valence electron of an alkaline atom or the electron in a hydrogen atom. Impose to the system a magnetic field  $\mathbf{B}(t)$  that has a constant magnitude  $B$  and rotates about some fixed axis at a constant angle  $\theta_B$  and with a constant frequency  $\omega$ . The rotating axis is chosen as the  $z$  axis of the coordinate system. So the magnetic field is

$$\mathbf{B}(t) = B\mathbf{n}(t), \quad \mathbf{n}(t) = (\sin \theta_B \cos \omega t, \sin \theta_B \sin \omega t, \cos \theta_B). \quad (1)$$

We take  $B > 0$  without loss of generality. The motion of the electron in the central potential of the nucleus (and the other electrons in the inner shells for alkaline atoms) and the above rotating magnetic field may be described by the Schrödinger equation

$$i\hbar\partial_t\psi = H(t)\psi, \quad (2a)$$

where

$$H(t) = H_0 + \mu_B B(\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}(t), \quad (2b)$$

where

$$H_0 = \frac{\mathbf{p}^2}{2M} + V(r) \quad (2c)$$

is the Hamiltonian of the electron when the external magnetic field is absent,  $M$  and  $\mu_B$  are respectively the reduced mass and the Bohr magneton of the electron,  $\mathbf{l} = \mathbf{r} \times \mathbf{p}/\hbar$  is the orbit angular momentum operator in unit of  $\hbar$ , and  $\mathbf{s}$  the spin in the same unit. Let us make some remarks on the above Hamiltonian  $H(t)$ . First, we are considering a strong magnetic field, so a spin-orbit coupling term is omitted. Second, the magnetic field is not too strong such that a quadratic term in  $\mathbf{B}(t)$  can be omitted as well. Omission of this term also implies that we would not consider very high excited states, especially scattering states, because the term is also quadratic in  $r$ . Third, the rotating magnetic field would generate a time varying electric field. Thus we are indeed dealing with a time varying electromagnetic field. In the nonrelativistic limit, however, the electric field does not enter the Schrödinger equation. What enters the equation is the vector potential  $A_\mu$ , which has been taken as

$$\mathbf{A}(t) = \frac{1}{2}\mathbf{B}(t) \times \mathbf{r}, \quad (3)$$

and  $-eA_0 = V(r)$ , where we denote the charge of the electron as  $-e$ . The vector potential  $\mathbf{A}(t)$  produces the magnetic field (1) and results in the Eq. (2). It also produces the above mentioned electric field. When Eq. (2) is derived from the relativistic Dirac equation, a term of the form  $\boldsymbol{\alpha} \cdot \mathbf{E}$  has been discarded, where  $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$  and the  $\gamma$ 's are the Dirac matrices. In this term  $\mathbf{E}$  contains two parts, one from  $V(r)$  and the other from  $\mathbf{A}(t)$ . The contribution to the Hamiltonian from the first part can be neglected in comparison with  $V(r)$ , as far as bound states are concerned. The contribution from the second part can be neglected in comparison with the second term in Eq. (2b) for bound states if  $\omega \ll 10^{18}$  Hz (by a rough estimation) which imposes no practical restriction on  $\omega$ . Therefore one can indeed forget the existence of the time varying electric field for the situation being considered.

Now we are going to solve the equation (2). We make a time-dependent unitary transformation

$$\psi(t) = W(t)\Phi(t), \quad (4)$$

where

$$W(t) = \exp(-i\omega t j_z), \quad (5)$$

where  $j_z$  is the  $z$ -component of the total angular momentum (in unit of  $\hbar$ )

$$\mathbf{j} = \mathbf{l} + \mathbf{s}. \quad (6)$$

This transformation is a generalization of the one used in solving the Schrödinger equation for a neutral particle with arbitrary spin in the rotating magnetic field [2]. It is not difficult to show that

$$W^\dagger(t)H(t)W(t) = H(0), \quad (7)$$

where

$$H(0) = H_0 + \mu_B B(\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}_0, \quad (8)$$

where  $\mathbf{n}_0 = \mathbf{n}(0) = (\sin \theta_B, 0, \cos \theta_B)$ . Thus we obtain the following equation for  $\Phi$ :

$$i\hbar \partial_t \Phi = H_{\text{eff}} \Phi, \quad (9)$$

where the effective Hamiltonian

$$H_{\text{eff}} = H(0) - \hbar \omega j_z = H_0 + \hbar \omega_0(\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}_0 - \hbar \omega(l_z + s_z) \quad (10)$$

is time independent. In the above equation we have define  $\omega_0 = \mu_B B/\hbar$  which is positive. Now let us define two new frequencies

$$\omega_L = (\omega_0^2 + \omega^2 - 2\omega_0\omega \cos \theta_B)^{1/2}, \quad (11)$$

$$\omega_S = (4\omega_0^2 + \omega^2 - 4\omega_0\omega \cos \theta_B)^{1/2}, \quad (12)$$

and introduce two unit vectors

$$\mathbf{n}_L = (\sin \theta_L, 0, \cos \theta_L), \quad (13)$$

$$\mathbf{n}_S = (\sin \theta_S, 0, \cos \theta_S), \quad (14)$$

where

$$\sin \theta_L = \frac{\omega_0 \sin \theta_B}{\omega_L}, \quad \cos \theta_L = \frac{\omega_0 \cos \theta_B - \omega}{\omega_L}, \quad (15)$$

$$\sin \theta_S = \frac{2\omega_0 \sin \theta_B}{\omega_S}, \quad \cos \theta_S = \frac{2\omega_0 \cos \theta_B - \omega}{\omega_S}. \quad (16)$$

Then the effective Hamiltonian can be written as

$$H_{\text{eff}} = H_0 + \hbar \omega_L \mathbf{l} \cdot \mathbf{n}_L + \hbar \omega_S \mathbf{s} \cdot \mathbf{n}_S. \quad (17)$$

Therefore the effective direction of the magnetic field is different for the orbit and spin angular momentum. The eigenvalue problem of this effective Hamiltonian can be completely solved. This can be easily realized when it is recasted in the following form.

$$H_{\text{eff}} = \exp(-i\theta_L l_y - i\theta_S s_y)(H_0 + \hbar\omega_L l_z + \hbar\omega_S s_z) \exp(i\theta_L l_y + i\theta_S s_y). \quad (18)$$

We denote the common eigenstates of  $(H_0, \mathbf{l}^2, l_z)$  as  $\zeta_{nlm}^0$  and the eigenstates of  $s_z$  as  $\chi_{m_s}^0$ , the eigenvalues are respectively  $(\epsilon_{nl}, l(l+1), m)$  and  $m_s$ , where  $\epsilon_{nl}$  are the energy spectrum of the electron in the absence of the magnetic field. The eigenstates of  $H_{\text{eff}}$  are then

$$\varphi_{nlmm_s} = \zeta_{nlm} \chi_{m_s}, \quad (19)$$

where

$$\zeta_{nlm} = \exp(-i\theta_L l_y) \zeta_{nlm}^0, \quad \chi_{m_s} = \exp(-i\theta_S s_y) \chi_{m_s}^0. \quad (20)$$

The corresponding energy eigenvalues are

$$E_{nlmm_s} = \epsilon_{nl} + m\hbar\omega_L + m_s\hbar\omega_S. \quad (21)$$

There is no degeneracy in the quantum numbers. This is different from the case in a static magnetic field (Paschen-Back effect) where some degeneracy is preserved. Unfortunately, the above energy levels are not observable since  $H_{\text{eff}}$  is not a physical quantity. Note that  $\varphi_{nlmm_s}$  are also the common eigenstates of the operators  $(H_0, \mathbf{l}^2, \mathbf{l} \cdot \mathbf{n}_L, \mathbf{s} \cdot \mathbf{n}_S)$  with eigenvalues  $(\epsilon_{nl}, l(l+1), m, m_s)$ . Since  $H_{\text{eff}}$  is time-independent, Eq. (9) has the formal solution

$$\Phi(t) = U_{\text{eff}}(t)\Phi(0), \quad U_{\text{eff}}(t) = \exp(-iH_{\text{eff}}t/\hbar). \quad (22)$$

With the obvious relation  $\psi(0) = \Phi(0)$ , the time-dependent Schrödinger equation (2) is formally solved as

$$\psi(t) = U(t)\psi(0), \quad U(t) = W(t)U_{\text{eff}}(t). \quad (23)$$

Since  $U(t)$  involves no chronological product, this solution is convenient for practical calculations.

Now we show that eigenstates of the effective Hamiltonian correspond to cyclic solutions of Eq. (2). We take the initial condition

$$\psi_i(0) = \varphi_i = \varphi_{nlmm_s}, \quad (24)$$

and calculate  $\psi_i(T)$  where  $T = 2\pi/\omega$  is the period of the rotating magnetic field. Here for convenience we use one subscript  $i$  to represent all the quantum numbers  $nlmm_s$ . Obviously,  $U_{\text{eff}}(t)\varphi_i = \exp(-iE_i t/\hbar)\varphi_i$ , valid for all  $t$ , and  $W(T)\varphi_i = \exp(-i2\pi l_z)\zeta_{nlm} \exp(-i2\pi s_z)\chi_{m_s}$ . Because

$$\zeta_{nlm} = \sum_{m'} D_{m'm}^l(0, \theta_L, 0) \zeta_{nlm'}^0, \quad \chi_{m_s} = \sum_{m'_s} D_{m'_s m_s}^{1/2}(0, \theta_S, 0) \chi_{m'_s}^0, \quad (25)$$

where the  $D$ 's are Wigner functions, we have  $W(T)\varphi_i = \exp(-i2\pi m - i2\pi m_s)\varphi_i$ . Finally we obtain

$$\psi_i(T) = \exp(-iE_i T/\hbar - i2\pi m - i2\pi m_s)\psi_i(0). \quad (26)$$

Hence it is indeed a cyclic solution, and the total phase change in a period is

$$\delta_i = -E_i T/\hbar - 2\pi m - 2\pi m_s = -E_i T/\hbar - \pi, \quad \text{mod } 2\pi. \quad (27)$$

To determine the dynamic phase, we should calculate

$$\langle H(t) \rangle_i \equiv (\psi_i(t), H(t)\psi_i(t)) = (\psi_i(0), W^\dagger H(t)W\psi_i(0)) = (\psi_i(0), H(0)\psi_i(0)).$$

Because  $H(0) = H_{\text{eff}} + \hbar\omega j_z$ , we have

$$\langle H(t) \rangle_i = E_i + \hbar\omega \langle j_z \rangle_i. \quad (28)$$

Here  $\langle j_z \rangle_i = (\psi_i(0), j_z \psi_i(0)) = (\psi_i(t), j_z \psi_i(t))$  is the expectation value of  $j_z$  in the state  $\psi_i(t)$ , and it is time independent. Note that  $\langle H(t) \rangle_i$  is also independent of  $t$ . Thus the state  $\psi_i(t)$  is somewhat similar to a stationary state in a system with a time-independent Hamiltonian. The dynamic phase is

$$\beta_i = -\hbar^{-1} \int_0^T dt \langle H(t) \rangle_i = -E_i T/\hbar - 2\pi \langle j_z \rangle_i. \quad (29)$$

Therefore the nonadiabatic geometric phase is

$$\gamma_i = \delta_i - \beta_i = -\pi + 2\pi \langle j_z \rangle_i, \quad \text{mod } 2\pi, \quad (30)$$

and is determined by  $\langle j_z \rangle_i$ . This is a generalization of the result for a neutral particle [2]. For the above calculations to stand, only two facts are necessary. First, the initial state is an eigenstate of  $H_{\text{eff}}$ . Second, it can be expanded as a linear combination of the common eigenstates of  $(l_z, s_z)$ . The latter is independent of the above specific form of  $H_{\text{eff}}$ . Therefore the result (30) is valid regardless of whether  $\varphi_i$  is explicitly available or not, and is convenient for approximate calculations if necessary.

For the present case, the above geometric phase can be worked out explicitly. Indeed, it is easy to show that

$$\exp(i\theta_L l_y) l_z \exp(-i\theta_L l_y) = -\sin \theta_L l_x + \cos \theta_L l_z,$$

thus in the  $i$ th cyclic solution with  $\psi_i(0) = \varphi_i$ , we have

$$\langle l_z \rangle_i = (\varphi_i, l_z \varphi_i) = (\zeta_{nlm}, l_z \zeta_{nlm}) = (\zeta_{nlm}^0, \exp(i\theta_L l_y) l_z \exp(-i\theta_L l_y) \zeta_{nlm}^0) = m \cos \theta_L. \quad (31)$$

Similarly, we have

$$\langle s_z \rangle_i = m_s \cos \theta_S. \quad (32)$$

Therefore the geometric phase takes the form

$$\gamma_i = -m\Omega_L - m_s\Omega_S, \quad \text{mod } 2\pi, \quad (33)$$

where

$$\Omega_L = 2\pi(1 - \cos \theta_L), \quad \Omega_S = 2\pi(1 - \cos \theta_S) \quad (34)$$

are the solid angles subtended by the traces of orbit and spin angular momentum. Indeed, it is easy to show that

$$(\psi_i(t), \mathbf{l}\psi_i(t)) = m(\sin \theta_L \cos \omega t, \sin \theta_L \sin \omega t, \cos \theta_L), \quad (35)$$

$$(\psi_i(t), \mathbf{s}\psi_i(t)) = m_s(\sin \theta_S \cos \omega t, \sin \theta_S \sin \omega t, \cos \theta_S). \quad (36)$$

Therefore the geometric nature of the results is quite obvious. It should be remarked that the orbit and spin angular momentum both precess synchronously with the magnetic field, but each at a different angle with the rotating axis.

In the above paragraph, we have shown that the initial condition  $\psi_i(0) = \varphi_i$  leads to a cyclic solution. On the other hand, a cyclic solution need not always take such an initial condition. If the parameters are appropriately chosen, one can find other cyclic solutions explicitly. For example, if  $B$  (or  $\omega_0$ ) and  $\theta_B$  are chosen such that

$$\omega_L = N_L \omega, \quad \omega_S = N_S \omega, \quad (37)$$

where  $N_L$  and  $N_S$  are natural numbers, then any solution with the initial condition

$$\psi(0) = \sum_{mm_s} a_{mm_s} \varphi_{nlmm_s} \quad (38)$$

is a cyclic solution, where the coefficients  $a_{mm_s}$  are arbitrary. In fact, one can show that

$$\psi(T) = \exp[-i\epsilon_{nl}T/\hbar - i(N_S + 1)\pi]\psi(0). \quad (39)$$

To conclude this section we give a specific realization of the condition (37). If one chooses  $\omega_0/\omega = \sqrt{3/2}$  and  $\cos \theta_B = \sqrt{3}/2\sqrt{2}$ , then the condition (37) holds with  $N_L = 1$ ,  $N_S = 2$ .

### III. ALKALINE ATOMIC ELECTRON IN A WEAK ROTATING MAGNETIC FIELD

In this section we consider the same problem as in Sec. II but with a weak rotating magnetic field. In this case a spin-orbit coupling term has to be included in the Hamiltonian, and the Schrödinger equation reads

$$i\hbar\partial_t\psi = H(t)\psi, \quad (40a)$$

where

$$H(t) = H_0 + \mu_B B(\mathbf{l} + 2\mathbf{s}) \cdot \mathbf{n}(t) + \hbar^2 \xi(r) \mathbf{l} \cdot \mathbf{s}. \quad (40b)$$

We make use of the time-dependent unitary transformation (4-5). Since the spin-orbit coupling term is invariant under the transformation, Eq. (7) holds for the present case. Thus we get for  $\Phi$  the equation

$$i\hbar\partial_t\Phi = H_{\text{eff}}\Phi, \quad (41)$$

where the effective Hamiltonian is now given by

$$H_{\text{eff}} = H_0 + \hbar\omega_L \mathbf{l} \cdot \mathbf{n}_L + \hbar\omega_S \mathbf{s} \cdot \mathbf{n}_S + \hbar^2\xi(r)\mathbf{l} \cdot \mathbf{s}. \quad (42)$$

As in Sec. II,  $H_{\text{eff}}$  is time independent, so we have the formal solutions for Eqs. (41) and (40)

$$\Phi(t) = U_{\text{eff}}(t)\Phi(0), \quad U_{\text{eff}}(t) = \exp(-iH_{\text{eff}}t/\hbar), \quad (43)$$

$$\psi(t) = U(t)\psi(0), \quad U(t) = W(t)U_{\text{eff}}(t). \quad (44)$$

Because  $H_{\text{eff}}$  is an Hermitian operator, one can in principle find a complete set of eigenstates  $\{\varphi_i\}$ , with eigenvalues  $\{E_i\}$ . A cyclic solution of the Schrödinger equation can be obtained by taking anyone of these eigenstates as an initial state, i.e.,  $\psi_i(0) = \varphi_i$ . In fact, one can always expand  $\varphi_i$  as a linear combination of the common eigenstates of  $(l_z, s_z)$ , and obtain

$$\psi_i(T) = \exp(-iE_iT/\hbar - i\pi)\psi_i(0). \quad (45)$$

The geometric phase of this solution in a period can be calculated in a way similar to that in Sec. II. The result has the same form:

$$\gamma_i = -\pi + 2\pi\langle j_z \rangle_i, \quad \text{mod } 2\pi. \quad (46)$$

Unfortunately, in the present case the eigenvalue problem of  $H_{\text{eff}}$  is difficult to solve in general. The only operator that commute with  $H_{\text{eff}}$  and thus can have common eigenstates is  $\mathbf{l}^2$ . (Of course  $\mathbf{s}^2$  is another such operator, but it is trivial.) We may denote these common eigenstates as  $\varphi_i = \varphi_{lp}$  where  $p$  represents all other quantum numbers. Here we deal only with the special case  $l = 0$  where the eigenstates are explicitly available. They read

$$\varphi_{0p} = \zeta_{n00}^0 \exp(-i\theta_S s_y) \chi_{m_s}^0, \quad (47)$$

where  $\zeta_{n00}^0$  and  $\chi_{m_s}^0$  are those used in Sec. II. In this case the geometric phase can be easily found to be

$$\gamma_{0p} = -m_s\Omega_S, \quad \text{mod } 2\pi. \quad (48)$$

#### IV. GENERAL CHARGED PARTICLES IN A ROTATING MAGNETIC FIELD

In this section we consider a general charged particle with an arbitrary spin  $s$  ( $s = 1/2, 1, 3/2, \dots$ ) and with magnetic moment moving in the rotating magnetic field (1), without a central potential. We denote the charge of the particle by  $q$ , the mass by  $M$ , the spin angular momentum by  $\mathbf{s}$ , and the magnetic moment by  $\mu$ . In this case the Schrödinger equation reads

$$i\hbar\partial_t\psi = H(t)\psi, \quad (49a)$$



where

$$H(t) = \frac{1}{2M} \left[ \mathbf{p} - \frac{q}{c} \mathbf{A}(t) \right]^2 - \boldsymbol{\mu} \cdot \mathbf{B}(t), \quad (49b)$$

where  $\boldsymbol{\mu} = \mu \mathbf{s}/s$ , and  $\mathbf{A}(t)$  is given by Eq. (3), which produces the magnetic field. Note that  $\mathbf{A}(t)$  also produces an time-dependent electric field as pointed out before. Now that the charged particle is not necessarily confined in a small region, there seems no reason to neglect the existence of the electric field. Thus we are actually dealing with an electromagnetic field. Fortunately, the electric field does not enter the equation as long as we confine ourselves to a nonrelativistic theory. This is also the reason why we only speak about the magnetic field.

The above Hamiltonian can be rewritten in the form

$$H(t) = \frac{\mathbf{p}^2}{2M} + \frac{1}{2} M \omega_1^2 [r^2 - (\mathbf{r} \cdot \mathbf{n}(t))^2] - \epsilon(q) \hbar \omega_1 \mathbf{l} \cdot \mathbf{n}(t) - \epsilon(\mu) \hbar \omega_2 \mathbf{s} \cdot \mathbf{n}(t), \quad (50)$$

where  $\omega_1 = |q|B/2Mc$ ,  $\omega_2 = |\mu|B/s\hbar$ , both being positive, and  $\epsilon(q)$ ,  $\epsilon(\mu)$  are sign functions. As before, we make use of the time-dependent unitary transformation (4-5). It can be shown that  $W^\dagger(t)H(t)W(t) = H(0)$  still holds in the present case. Thus we obtain the following equation for  $\Phi$ :

$$i\hbar \partial_t \Phi = H_{\text{eff}} \Phi, \quad (51)$$

where

$$H_{\text{eff}} = H(0) - \hbar \omega j_z,$$

or

$$H_{\text{eff}} = \frac{\mathbf{p}^2}{2M} + \frac{1}{2} M \omega_1^2 [r^2 - (\mathbf{r} \cdot \mathbf{n}_0)^2] - \epsilon(q) \hbar \omega_1 \mathbf{l} \cdot \mathbf{n}_0 - \epsilon(\mu) \hbar \omega_2 \mathbf{s} \cdot \mathbf{n}_0 - \hbar \omega (l_z + s_z). \quad (52)$$

As before,  $H_{\text{eff}}$  is time independent, so we have the formal solutions for Eqs. (51) and (49)

$$\Phi(t) = U_{\text{eff}}(t) \Phi(0), \quad U_{\text{eff}}(t) = \exp(-iH_{\text{eff}}t/\hbar), \quad (53)$$

$$\psi(t) = U(t) \psi(0), \quad U(t) = W(t) U_{\text{eff}}(t). \quad (54)$$

Because  $H_{\text{eff}}$  is an Hermitian operator, one can in principle find a complete set of eigenstates  $\{\varphi_i\}$ , with eigenvalues  $\{E_i\}$ . A cyclic solution of the Schrödinger equation can be obtained by taking anyone of these eigenstates as an initial state, i.e.,  $\psi_i(0) = \varphi_i$ . In fact, one can show that

$$\psi_i(T) = \exp(-iE_iT/\hbar - i2\pi s) \psi_i(0). \quad (55)$$

The geometric phase of this solution in a period can be found to be

$$\gamma_i = -2\pi s + 2\pi \langle j_z \rangle_i, \quad \text{mod } 2\pi. \quad (56)$$

These are all familiar results. The remaining task is to find the eigenvalues and eigenstates of  $H_{\text{eff}}$ , and to work out the geometric phase explicitly.

Since the effective Hamiltonian is somewhat complicated, we have to make some approximation. We assume that  $\omega \ll \omega_1$ ,  $\omega \ll \omega_2$ , that is, we are considering slowly rotating magnetic field. This is in fact a rather loose restriction. To see this, let us make some rough estimation. For a magnetic field of the magnitude 1 tesla (which is not very large), we have for an electron  $\omega_1 \sim 10^{10}$  Hz,  $\omega_2 = 2\omega_1$ , and for a proton  $\omega_1 \sim 10^7$  Hz,  $\omega_2 \sim 10^8$  Hz. An experimentally available rotating frequency must be much smaller than these ones. Under this assumption, the term  $\hbar\omega l_z$  in Eq. (52) is much smaller than the two preceding terms and thus can be treated as a small perturbation. On the other hand, the term  $\hbar\omega s_z$ , which is of the same order of magnitude, need not be treated approximately. We thus decompose  $H_{\text{eff}}$  as

$$H_{\text{eff}} = H_{\text{eff}}^0 + H'_{\text{eff}}, \quad (57)$$

where

$$H_{\text{eff}}^0 = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega_1^2[r^2 - (\mathbf{r} \cdot \mathbf{n}_0)^2] - \epsilon(q)\hbar\omega_1 \mathbf{l} \cdot \mathbf{n}_0 - \epsilon(\mu)\hbar\omega_S \mathbf{s} \cdot \mathbf{n}_S, \quad (58)$$

whose eigenvalue problem will be solved exactly, and

$$H'_{\text{eff}} = -\hbar\omega l_z \quad (59)$$

will be treated as a small perturbation. In Eq. (58),

$$\omega_S = [\omega_2^2 + \omega^2 + 2\epsilon(\mu)\omega_2\omega \cos \theta_B]^{1/2}, \quad (60)$$

$$\mathbf{n}_S = (\sin \theta_S, 0, \cos \theta_S), \quad (61)$$

where

$$\sin \theta_S = \frac{\omega_2 \sin \theta_B}{\omega_S}, \quad \cos \theta_S = \frac{\omega_2 \cos \theta_B + \epsilon(\mu)\omega}{\omega_S}. \quad (62)$$

It is not difficult to show that

$$H_{\text{eff}}^0 = \exp(-i\theta_B l_y - i\theta_S s_y) H_{\text{eff}}^z \exp(i\theta_B l_y + i\theta_S s_y), \quad (63)$$

where

$$H_{\text{eff}}^z = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega_1^2 \rho^2 - \epsilon(q)\hbar\omega_1 l_z - \epsilon(\mu)\hbar\omega_S s_z \quad (64)$$

is the Hamiltonian of a charged particle in a static magnetic field along the  $z$  axis (but note that  $\omega_2$  is replaced by  $\omega_S$ ). In the above equation  $\rho$  is a cylindrical coordinate. The eigenvalue problem of  $H_{\text{eff}}^z$  is well known. In the cylindrical coordinates, the eigenfunctions are

$$u_i = u_{n_\rho n_z m m_s}(\rho, \phi, z, s_z) = N_{n_\rho m} \exp\left(-\frac{1}{2}\alpha^2 \rho^2\right) (\alpha\rho)^{|m|} L_{n_\rho}^{|m|}(\alpha^2 \rho^2) \frac{e^{im\phi}}{\sqrt{2\pi}} \frac{\exp(i2\pi n_z z/d)}{\sqrt{d}} \chi_{m_s}^0, \quad (65)$$

where  $m$  and  $n_z$  take on integer values,  $n_\rho = 0, 1, 2, \dots$ ,  $m_s = s, s-1, \dots, -s$ ,  $\alpha = \sqrt{M\omega_1/\hbar}$ ,  $d$  is a length in the  $z$  direction for box normalization, the  $L_{n_\rho}^{|m|}$  are Laguerre polynomials [9], and the normalization constant

$$N_{n_\rho m} = \alpha \left[ \frac{2n_\rho!}{\Gamma(n_\rho + |m| + 1)} \right]^{1/2}.$$

Note that these are also common eigenstates of  $(l_z, s_z)$  with eigenvalues  $(m, m_s)$ . The eigenvalue of  $H_{\text{eff}}^z$  is

$$E_i^0 = E_{n_\rho n_z m m_s}^0 = (2n_\rho + |m| + 1)\hbar\omega_1 + \frac{2n_z^2\pi^2\hbar^2}{Md^2} - \epsilon(q)m\hbar\omega_1 - \epsilon(\mu)m_s\hbar\omega_S. \quad (66)$$

There are some degeneracy in the quantum numbers  $m$  and  $n_\rho$ . The eigenfunctions of  $H_{\text{eff}}^0$  are given by

$$\varphi_i^0 = \exp(-i\theta_B l_y - i\theta_S s_y) u_i. \quad (67)$$

The corresponding energy eigenvalues are still given by Eq. (66). We will use these  $\varphi_i^0$  as the approximate eigenfunctions of  $H_{\text{eff}}$ . The lowest-order corrections to the energy eigenvalues are given by the expectation values of  $H'_{\text{eff}}$  in these eigenstates. The corrected energy levels are

$$E_i = E_{n_\rho n_z m m_s} = (2n_\rho + |m| + 1)\hbar\omega_1 + \frac{2n_z^2\pi^2\hbar^2}{Md^2} - m\hbar[\epsilon(q)\omega_1 + \omega \cos \theta_B] - \epsilon(\mu)m_s\hbar\omega_S. \quad (68)$$

Now there is no degeneracy in the quantum numbers. The geometric phase of the  $i$ th state in a period is approximately given by

$$\gamma_i = -m\Omega_B - m_s\Omega_S, \quad \text{mod } 2\pi, \quad (69)$$

where  $\Omega_S$  has been defined in Eq. (34), and  $\Omega_B = 2\pi(1 - \cos \theta_B)$ . Note that the second term is exact. The approximation lies in the first term.

Though the above  $u_i$  or  $\varphi_i^0$  are not eigenfunctions of  $\mathbf{I}^2$ , it can still be shown that

$$(\psi_i(t), \mathbf{l}\psi_i(t)) = m(\sin \theta_B \cos \omega t, \sin \theta_B \sin \omega t, \cos \theta_B), \quad (70)$$

$$(\psi_i(t), \mathbf{s}\psi_i(t)) = m_s(\sin \theta_S \cos \omega t, \sin \theta_S \sin \omega t, \cos \theta_S). \quad (71)$$

Therefore the orbit and spin angular momentum both precess synchronously with the magnetic field, the former at an angle  $\theta_B$  (approximate) and the latter at  $\theta_S$  (exact) with the rotating axis. The geometric nature of the result (69) is thus obvious.

## V. SUMMARY AND DISCUSSIONS

In this paper we consider a nonrelativistic charged particle moving in a rotating magnetic field, with or without a central potential. The case with a central potential may describe the

valence electron in an alkaline atom or the only one in a hydrogen atom. The Hamiltonian for such a system is time dependent. By making use of a time-dependent unitary transformation, the Schrödinger equation can be reduced to a Schrödinger-like equation with an effective Hamiltonian which is time independent. In this way one obtains a formal solution to the original Schrödinger equation, which determines the time evolution of an arbitrary initial state. The time-evolution operator in this formal solution, unlike the one for a general time-dependent Hamiltonian, involves no chronological product, and thus is convenient for practical calculations. Any solution with one of the eigenstates of the effective Hamiltonian as an initial state is a cyclic solution. The geometric phase in a period for such a solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis. This holds regardless of whether the solution is explicitly available, and is convenient for approximate calculations whenever necessary. We discuss in detail the alkaline atomic electron in a strong or weak magnetic field. In a strong magnetic field everything is worked out explicitly. In a weak field, however, the effective Hamiltonian is complicated and the eigenvalue problem of it is explicitly solved only in a special case. For a general charged particle and without the central potential, the eigenvalue problem of the effective Hamiltonian is solved approximately. It is a rather good approximation if the magnetic field rotates slowly.

The time-dependent unitary transformation used in this paper is a generalization of the one previously used for a neutral particle with arbitrary spin and with magnetic moment in a rotating magnetic field [2]. It transforms the Schrödinger equation to one in a rotating frame where the magnetic field is static. Thus in that frame the effective Hamiltonian is time independent. An extended rotating-frame formalism has been used to study a neutral particle in a more general time-dependent magnetic field [10]. It should be remarked, however, that the transformation is merely a mathematical technique, and the effective equation in the rotating frame does not describe a real physical problem whose results are observable in that frame. A real physical problem in a rotating frame would be much more complicated since the frame is not an inertial system.

The extension of the present problem to relativistic particles, either electrically charged or neutral, is currently under investigation. Progress will be reported subsequently.

## ACKNOWLEDGMENTS

The author is grateful to Professor Guang-jiong Ni for communications and encouragement. This work was supported by the National Natural Science Foundation of China.

- 
- [1] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1977).
- [2] S.-J. Wang, Phys. Rev. A **42**, 5107 (1990).
- [3] A. G. Wagh and V. C. Rakhecha, Phys. Lett. A **170**, 71 (1992).
- [4] G.-J. Ni, S.-Q. Chen, and Y.-L. Shen, Phys. Lett. A **197**, 100 (1995).
- [5] G.-J. Ni and S.-Q. Chen, *Advanced Quantum Mechanics* (Fudan Univ. Press, Shanghai, 2000) (in Chinese).
- [6] M. V. Berry, Proc. R. Soc. Lond. A **392**, 45 (1984).
- [7] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
- [8] H.-Z. Li, *Global Properties of Simple Physical Systems—Berry's Phase and Others* (Shanghai Scientific & Technical, Shanghai, 1998) (in Chinese).
- [9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1980).
- [10] A. G. Wagh and V. C. Rakhecha, Phys. Rev. A **48**, R1729 (1993).